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Homogenization of the planar waveguide with frequently alternating boundary conditions

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Abstract
We consider the Laplacian in a planar strip with a Dirichlet boundary condition on the upper boundary and with a frequent alternation boundary condition on the lower boundary. The alternation is introduced by the periodic partition of the boundary into small segments on which Dirichlet and Neumann conditions are imposed in turns. We show that under certain conditions the homogenized operator is the Dirichlet Laplacian and prove the uniform resolvent convergence. The spectrum of the perturbed operator consists of its essential part only and has a band structure. We construct the leading terms of the asymptotic expansions for the first band functions. We also construct the complete asymptotic expansion for the bottom of the spectrum.

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1. Introduction

The model of quantum waveguides with window(s) was studied in a series of papers by several authors; see [Bo3, BGRS, DK, ESTV, EV, G2, HTWK, Bo6, BEG]. Such waveguides were modeled by a pair of two planar strips or three-dimensional layers having a common boundary and with window(s) openings of finite size in it. The usual operator is the Dirichlet Laplacian. The main interest is the behavior of the spectrum of such an operator and its dependence on the window. If the strips or layers are of the same width, then the problem reduces to the Laplacian in one strip, and the window is modeled by segment(s) on the boundary where the Dirichlet condition switches to the Neumann one. This model poses interesting mathematical questions, and it is also of physical interest, since it has certain applications in nanophysical devices and in modeling electromagnetic waveguides.

It was shown in the above-cited papers that the perturbation by a finite number of the windows leaves the essential spectrum unchanged and gives rise to new discrete eigenvalues emerging below the threshold of the essential spectrum. This phenomenon was studied, and the behavior of the emerging eigenvalues was described.

A completely different situation occurs, if one deals with an infinite number of the windows located on the boundary. In this case, the perturbation is not localized and as a result it changes an essential spectrum. Exactly this situation is considered in this paper. Namely, we consider a planar strip with periodically located windows of the same length. The windows are modeled by segments where the Dirichlet boundary condition is replaced by the Neumann one. The main feature is that the sizes of these windows are small, and the distance between each two neighboring windows is small, too. Such a perturbation is well known in homogenization theory; see, for instance, [Bo4, FHY, C, Bo7, Bo5, BLP, DT]. In the case of the bounded domains, it is known that under certain conditions on the alternation the homogenized operator is the Dirichlet Laplacian, i.e., the homogenized boundary condition is the Dirichlet one. The same phenomenon occurs in our problem. In other words, in the limit the perturbed operator behaves as if there are no windows at all. Moreover, it happens even in the case when the size of the windows is relatively larger than the remaining parts with the Dirichlet condition; see condition (2.1) and theorem 2.1.

The above-mentioned convergence of the perturbed operator is in the uniform resolvent sense. It also holds true, if we consider the resolvent not only as an operator in L_2 but as those from L_2 into W_2^1. We give an effective estimate for the rate of the convergence. Such kinds of estimates for the operators with fast oscillating coefficients were obtained recently in the series of papers [BS2, BS3, Bo2, Z1, Z2, ZPT, PT, P]. Although the perturbation by fast oscillating coefficients is also typical for the homogenization theory and it has a number of features similar to the perturbation by frequent alternation of the boundary condition, in our case the situation is rather different from that in the cited papers. Namely, while considering the resolvent as an operator from L_2 into W_2^1, they had to introduce a special corrector to get an estimate for the rate of convergence. In our case, we do not need such a corrector, and the estimate for the rate of the convergence can be obtained in a rather easy way exactly for the difference of the resolvents. This is a specific feature of the problems of boundary homogenization and it was known before in the homogenization of the fast oscillating boundary in the case of a bounded domain; see [OSI, chapter III, section 4.1].

One more result of our paper concerns the behavior of the spectrum of the perturbed operator. The spectrum has the band structure and we describe the asymptotic behavior for the first band functions w.r.t. a small parameter. It implies that the length of the first band tends to infinity w.r.t. a small parameter, and therefore all possible gaps 'run' to infinity. We prove that the bottom of the spectrum corresponds to a periodic eigenfunction of the operator obtained by Floquet decomposition of the periodic operator. On the basis of this fact we obtain the complete asymptotic expansion of the bottom of the spectrum.

In conclusion, we describe briefly the contents of the paper. In the following section, we formulate the problem and give the main results. In the third section, we prove the uniform resolvent convergence of the perturbed operator. The fourth section is devoted to a similar result but for the operator on a periodicity cell obtained in the Floquet decomposition. In the last, fifth, section, we analyze the bottom of the spectrum of the perturbed operator.

2. Formulation of the problem and the main results

Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$, ε be a small positive parameter, $\eta = \eta(\varepsilon)$ be a function satisfying the estimate

$$0 < \eta(\varepsilon) < \frac{\pi}{2}$$

for all ε. We partition the real axis into two subsets,

$$\gamma_\varepsilon := \{x : |x_1 - \varepsilon\pi m| < \varepsilon\eta,\ m \in \mathbb{Z}, x_2 = 0\}, \qquad \Gamma_\varepsilon := Ox_1 \setminus \overline{\gamma}_\varepsilon.$$

By Ω, Γ_+, and Γ_- we denote the strip $\{x : 0 < x_2 < \pi\}$ and its upper and lower boundaries, respectively.

The main object of our study is the Laplacian in $L_2(\Omega)$ subject to the Dirichlet boundary condition on $\Gamma_+ \cup \gamma_\varepsilon$ and to the Neumann one on Γ_ε. Rigorously we introduce it as the self-adjoint operator in $L_2(\Omega)$ associated with a sesquilinear form:

$$\mathfrak{h}_\varepsilon[u, v] := (\nabla u, \nabla v)_{L_2(\Omega)} \quad \text{on} \quad \mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon),$$

where $\mathring{W}_2^1(Q, S)$ indicates the subset of the functions in $W_2^1(Q)$ having zero trace on the curve S. We will employ the symbol $\mathcal{H}_\varepsilon$ to denote this operator.

Remark 2.1. Although it is not one of the main issues of our paper, it is possible to describe explicitly the structure of the functions in the domain of $\mathcal{H}_\varepsilon$. More precisely, it is possible to describe their behavior at the end points of γ_ε. We refer to lemma 3.1 for more details.

The main aim of the paper is to study the behavior of the resolvent and of the spectrum of $\mathcal{H}_\varepsilon$ as $\varepsilon \to +0$. We introduce one more self-adjoint operator $\mathcal{H}_0$ which is the Dirichlet Laplacian in $L_2(\Omega)$. We define it as associated with a sesquilinear form:

$$\mathfrak{h}_0[u, v] := (\nabla u, \nabla v)_{L_2(\Omega)} \quad \text{on} \quad \mathring{W}_2^1(\Omega, \partial\Omega).$$

It is well known that the domain of this operator is $W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega, \partial\Omega)$. In what follows the symbol $\|\cdot\|_{A\to B}$ indicates the norm of an operator from the space A to B.

Our first result says that under the condition

$$\varepsilon \ln \eta(\varepsilon) \to 0, \qquad \varepsilon \to +0, \tag{2.1}$$

the operator $\mathcal{H}_0$ is the homogenized one for $\mathcal{H}_\varepsilon$.

Theorem 2.1. *Suppose (2.1). Then the estimate*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant \sqrt{13}\,\varepsilon^{1/4} |\ln \sin \eta(\varepsilon)|^{1/4} \tag{2.2}$$

holds true.

As it follows from (2.1), the quantity $\varepsilon|\ln \sin \eta(\varepsilon)|$ tends to zero as $\varepsilon \to +0$. Even if η tends to zero not very fast, say, as $\eta \sim \varepsilon^\alpha$, $\alpha > 0$, and the lengths of the Dirichlet parts on Γ_- are, therefore, relatively small with respect to those of the Neumann parts, the homogenized operator is still subject to the Dirichlet condition on Γ_-. This fact was known in the case of bounded domains; see, for instance, [FHY, C]. Moreover, if $\eta \to \frac{\pi}{2} - 0$ as $\varepsilon \to +0$, then the measures of Neumann parts of the boundary are relatively small w.r.t. to those of Dirichlet parts. In this case, $|\ln \sin \eta| \to +0$ and it improves the rate of convergence in (2.2).

The spectrum of $\mathcal{H}_0$ consists only of its essential component and coincides with the semi-axis $[1, +\infty)$. As a corollary of theorem 2.1, we have

Theorem 2.2. *The spectrum of $\mathcal{H}_\varepsilon$ converges to that of $\mathcal{H}_0$. Namely, if $\lambda \notin [1, +\infty)$, then $\lambda \notin \sigma(\mathcal{H}_\varepsilon)$ for ε small enough. And if $\lambda \in [1, +\infty)$, then there exists $\lambda_\varepsilon \in \sigma(\mathcal{H}_\varepsilon)$ so that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to +0$.*

The operator $\mathcal{H}_\varepsilon$ is a periodic one due to the periodicity of the sets γ_ε and Γ_ε, and its spectrum has a band structure. Namely, let

$$\Omega_\varepsilon := \left\{x : |x_1| < \tfrac{1}{2}\varepsilon\pi,\ 0 < x_2 < \pi\right\}, \qquad \mathring{\gamma}_\varepsilon := \partial\Omega_\varepsilon \cap \gamma_\varepsilon,$$
$$\mathring{\Gamma}_\varepsilon := \partial\Omega_\varepsilon \cap \Gamma_\varepsilon, \qquad \mathring{\Gamma}_\pm := \partial\Omega_\varepsilon \cap \Gamma_\pm.$$

By $\mathcal{H}_\varepsilon^{(p)}(\tau)$ we denote the self-adjoint operator in $L_2(\Omega_\varepsilon)$ associated with the sesquilinear form,

$$\mathring{\mathfrak{h}}_\varepsilon^{(p)}[u,v] := \left(\left(\mathrm{i}\frac{\partial}{\partial x_1}-\frac{\tau}{\varepsilon}\right)u,\left(\mathrm{i}\frac{\partial}{\partial x_1}-\frac{\tau}{\varepsilon}\right)v\right)_{L_2(\Omega_\varepsilon)}+\left(\frac{\partial u}{\partial x_2},\frac{\partial u}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)},$$

on $\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)$, where $\tau\in[-1,1)$. Here $\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)$ is the subset of the functions in $\mathring{W}^1_2(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)$ satisfying periodic boundary conditions on the lateral boundaries of Ω_ε. The operator $\mathcal{H}_\varepsilon^{(p)}(\tau)$ has a compact resolvent. To prove this, it is sufficient to reduce the equation $\mathcal{H}_\varepsilon^{(p)}(\tau)u-\lambda u=f$ to an operator equation of second kind in $\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)$ in a standard way; see, for instance, [L, chapter II, section 5] and [L, chapter II, section 5, remark 5.1]. The mentioned operator equation involves a compact operator that follows from the compact embedding of $\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)$ in $L_2(\Omega_\varepsilon)$. The last fact implies the compactness of the resolvent. Hence, the spectrum of $\mathcal{H}_\varepsilon^{(p)}(\tau)$ consists of a countably many discrete eigenvalues accumulating at infinity. We denote these eigenvalues by $\lambda_n(\tau,\varepsilon)$ and arrange them in the non-descending order with the multiplicity taking into account

$$\lambda_1(\tau,\varepsilon)\leqslant\lambda_2(\tau,\varepsilon)\leqslant\lambda_3(\tau,\varepsilon)\leqslant\cdots\leqslant\lambda_n(\tau,\varepsilon)\leqslant\cdots.$$

Let $\sigma(\cdot)$, $\sigma_{\mathrm{e}}(\cdot)$ be the spectrum and the essential spectrum of an operator. Then

$$\sigma(\mathcal{H}_\varepsilon)=\sigma_{\mathrm{e}}(\mathcal{H}_\varepsilon)=\bigcup_{n=1}^{\infty}\{\lambda_n(\tau,\varepsilon):\tau\in[-1,1)\}\tag{2.3}$$

that will be shown in lemma 4.1.

The rest of the results is devoted to the behavior of $\lambda_n(\tau,\varepsilon)$ as $\varepsilon\to+0$. First we establish a uniform resolvent convergence for $\mathcal{H}_\varepsilon^{(p)}(\tau)$.

By $\mathfrak{L}$ we denote the subspace of the functions in $L_2(\Omega_\varepsilon)$ which are independent of x_1, and we decompose $L_2(\Omega_\varepsilon)$ as follows:

$$L_2(\Omega_\varepsilon)=\mathfrak{L}\oplus\mathfrak{L}^\perp,$$

where $\mathfrak{L}^\perp$ indicates the orthogonal complement to $\mathfrak{L}$ in $L_2(\Omega_\varepsilon)$. In $\mathfrak{L}$, we introduce a self-adjoint operator $\mathcal{Q}$ as associated with a sesquilinear form:

$$\mathfrak{q}[u,v]:=\left(\frac{\mathrm{d}u}{\mathrm{d}x_2},\frac{\mathrm{d}v}{\mathrm{d}x_2}\right)_{L_2(0,\pi)}\quad\text{on}\quad\mathring{W}^1_2((0,\pi),\{0,\pi\}).$$

In other words, $\mathcal{Q}$ is the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x_2^2}$ in $L_2(0,\pi)$ subject to the Dirichlet boundary condition.

Theorem 2.3. *Let* $|\tau|<1-\delta$*, where* $0<\delta<1$ *is a fixed constant and assume (2.1). Then for sufficiently small* ε *the estimate*

$$\left\|\left(\mathcal{H}_\varepsilon^{(p)}(\tau)-\frac{\tau^2}{\varepsilon^2}\right)^{-1}-\mathcal{Q}^{-1}\oplus 0\right\|_{L_2(\Omega_\varepsilon)\to L_2(\Omega_\varepsilon)}\leqslant\frac{\varepsilon+5\varepsilon^{1/2}|\ln\sin\eta|^{1/2}}{\delta^{1/2}}$$

holds true.

The resolvent $\left(\mathcal{H}_\varepsilon^{(p)}(\tau)-\frac{\tau^2}{\varepsilon^2}\right)^{-1}$ is well defined that will be shown in the proof of lemma 4.2.

We should mention that the results of theorem 2.3 are close to those of theorem 1.2 in [FS]. Moreover, the technique we employ to prove theorem 2.3 is similar to that proposed in [FS]. The next theorem should be regarded as the corollary of theorem 2.3.

Theorem 2.4. *Let the hypothesis of theorem 2.3 holds. Then given any N there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$, $n \leqslant N$ *the eigenvalues* $\lambda_n(\tau, \varepsilon)$ *satisfy the asymptotics*

$$\begin{aligned} &\lambda_n(\tau,\varepsilon) = \frac{\tau^2}{\varepsilon^2} + n^2 + R_n(\tau,\varepsilon), \qquad \varepsilon \to +0, \\ &|R_n(\tau,\varepsilon)| \leqslant n^4 \frac{\sqrt{2}\varepsilon + 8\varepsilon^{1/2}|\ln\sin\eta|^{1/2}}{\delta^{1/2}}. \end{aligned} \tag{2.4}$$

The last theorem implies that the length of the first N bands of the spectrum $\{\lambda_n(\tau,\varepsilon) : \tau \in [-1, 1)\}$, $n = 1, \ldots, N$, are of order at least $\mathcal{O}(\varepsilon^{-2})$. Moreover, they overlap. It means that the first zone of the spectrum stretches as $\varepsilon \to +0$ and in the limit coincides with the semi-axis $[1, +\infty)$. It implies that all possible gaps in the spectrum of $\mathcal{H}_\varepsilon$ 'run' to infinity with the speed at least $\mathcal{O}(\varepsilon^{-2})$. This is a natural situation for the homogenization problems; see, for instance, [Bo1, B].

The bottom of the spectrum of $\mathcal{H}_\varepsilon$ is given by $\inf_{\tau\in[-1,1)} \lambda_1(\tau, \varepsilon)$, and by theorem 2.2 it converges to one as $\varepsilon \to +0$. The following theorem gives its complete asymptotic expansion as $\varepsilon \to +0$.

Theorem 2.5. *For* ε *small enough, the first eigenvalue* $\lambda_1(\tau, \varepsilon)$ *attains its infimum at* $\tau = 0$, *i.e.,*

$$\inf_{\tau\in[-1,1)} \lambda_1(\tau,\varepsilon) = \lambda_1(0,\varepsilon). \tag{2.5}$$

The asymptotics

$$\lambda_1(0,\varepsilon) = 1 + \sum_{j=1}^{\infty} \varepsilon^j \mu_j(\eta), \tag{2.6}$$

$$\mu_1(\eta) = \frac{2}{\pi}\ln\sin\eta(\varepsilon), \qquad \mu_2(\eta) = \frac{3}{\pi^2}\ln^2\sin\eta(\varepsilon), \tag{2.7}$$

holds true, and other μ_j *are determined in a recurrent way by (5.11). Moreover,*

$$\mu_j(\eta) = K_j \ln^j \eta + \mathcal{O}(\ln^{j-3}\eta), \qquad \eta \to +0, \tag{2.8}$$

where K_j *are some constants.*

We observe that due to (2.8) the coefficients μ_j has increasing logarithmic singularities as $\eta \to +0$. At the same time, the terms of the series (2.6) behave as $\mathcal{O}(\varepsilon^i \ln^i \eta)$, if $\eta \to +0$ as $\varepsilon \to +0$, and in view of condition (2.1) the series (2.6) remains an asymptotic one. We note that this phenomenon for the problems in the bounded domains with the frequent alternation of the boundary conditions was described first in [Bo4, Bo5].

Theorem 2.4 does not describe all the eigenvalues of the operator $\mathcal{H}_\varepsilon$. Namely, we conjecture that there exists a two-parametric family of the eigenvalues of $\mathcal{H}_\varepsilon$ behaving as

$$\lambda_{n,m}(\tau,\varepsilon) \sim \frac{(\tau+2m)^2}{\varepsilon^2} + n^2 + \cdots, m \in \mathbb{Z}, \qquad n \in \mathbb{N}.$$

The reason for such conjecture is that the right-hand side of this relation is, in fact, the eigenvalues of the operator:

$$\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)^2 - \frac{\partial^2}{\partial x_2^2}$$

in $L_2(\Omega_\varepsilon)$ subject to the Dirichlet boundary condition on $\mathring{\Gamma}_+ \cup \mathring{\Gamma}_-$ and to the periodic boundary condition on the lateral boundaries of Ω_ε. Such an operator appears, if one treats $\mathcal{H}_0$ as periodic w.r.t. x_1 and makes the Floquet decomposition. Moreover, it is natural to expect that the same formulae are valid not for $|\tau| < 1-\delta$, as in theorem 2.4, but for all $\tau \in [-1, 1)$. Such formulae would allow to answer one more interesting question in the presence or absence of the gaps in the spectrum of $\mathcal{H}_\varepsilon$. As we said above, if they exist, such gaps 'run' to infinity as $\varepsilon \to +0$. Generally speaking, the lengths of the gaps could be small, finite or infinite as $\varepsilon \to +0$. At the same time, in [Bo1] it was shown that the spectrum of a periodic one-dimensional Schrödinger,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\left(\frac{x}{\varepsilon}\right) \quad \text{in} \quad L_2(\mathbb{R}),$$

can contains only the gaps of finite or small lengths. So, it allows us to conjecture the same for $\mathcal{H}_\varepsilon$ provided the gaps exist.

3. Convergence of the resolvent of $\mathcal{H}_\varepsilon$

This section is devoted to the proof of theorems 2.1, 2.2.

Let $\chi = \chi(t) \in C^\infty(\mathbb{R})$ be a cut-off function with values in [0, 1] equalling one as $t < 1$ and vanishing as $t > 2$. By $\mathcal{D}(\cdot)$ we denote the domain of an operator.

We indicate by $\left(r_\pm^{(m)}, \theta_\pm^{(m)}\right)$ the polar coordinates centered at $(\varepsilon\pi m \pm \varepsilon\eta, 0)$, $m \in \mathbb{Z}$, so that $\theta_\pm^{(m)} = 0$ corresponds to the points of γ_ε.

Lemma 3.1. *Each function $u \in \mathcal{D}(\mathcal{H}_\varepsilon)$ can be represented as*

$$\begin{aligned} &u(x) = \overset{0}{u}(x) + \overset{1}{u}(x), \\ &\overset{0}{u}(x) = \sum_{m\in\mathbb{Z}} \alpha_\pm^{(m)} \sqrt{r_\pm^{(m)}}\, \chi\left(\frac{3r_\pm^{(m)}}{\varepsilon\delta_\varepsilon}\right) \sin\frac{\theta_\pm^{(m)}}{2}, \\ &\delta_\varepsilon := \min\left\{\eta(\varepsilon), \frac{\pi}{2} - \eta(\varepsilon)\right\}, \end{aligned} \tag{3.1}$$

where $\alpha_\pm^{(m)}$ are some constants, and $\overset{1}{u} \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$. The estimate

$$\sum_{m\in\mathbb{Z}} \left(|\alpha_+^{(m)}|^2 + |\alpha_-^{(m)}|^2\right) + \|\overset{1}{u}\|^2_{W_2^2(\Omega)} \leqslant C\|\mathcal{H}_\varepsilon u\|^2_{L_2(\Omega)} \tag{3.2}$$

holds true, where the constant C is independent of u.

Proof. The domain of $\mathcal{H}_\varepsilon$ consists of the generalized solutions $u \in W_2^1(\Omega)$ to the problem:

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma_+ \cup \gamma_\varepsilon, \qquad \frac{\partial u}{\partial x_2} = 0 \quad \text{on} \quad x \in \Gamma_\varepsilon.$$

It follows that

$$\mathfrak{h}_\varepsilon[u, u] = (f, u). \tag{3.3}$$

Since $u = 0$ on Γ_+, the first eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x_2^2}$ on the cross section of Ω is at least 1/4. This is why

$$\left\|\frac{\partial u}{\partial x_2}(x_1, \cdot)\right\|^2_{L_2(0,\pi)} \geqslant \frac{1}{4}\|u(x_1, \cdot)\|^2_{L_2(0,\pi)}. \tag{3.4}$$

Hence,

$$\mathfrak{h}_\varepsilon[u, u] \geqslant \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} \geqslant \frac{1}{4}\|u\|^2_{L_2(\Omega_\varepsilon)}, \qquad \mathcal{H}_\varepsilon \geqslant 1/4,$$

and it follows from (3.3) that

$$\|u\|_{L_2(\Omega)} \leqslant 4\|f\|_{L_2(\Omega)}, \qquad \|\nabla u\|_{L_2(\Omega)} \leqslant 2\|f\|_{L_2(\Omega)}.$$

Employing these inequalities and proceeding as in the proof of theorem 2.1 in [Bo3], one can prove easily the representation (3.1), and the estimates

$$\|\overset{1}{u}\|_{W_2^2(\Omega)} \leqslant C\|\mathcal{H}_\varepsilon u\|_{L_2(\Omega)},$$
$$\left|\alpha_\pm^{(m)}\right| \leqslant C\left(\|\mathcal{H}_\varepsilon u\|_{L_2(\{x\in\Omega:\, r_\pm^{(m)}<\varepsilon\delta_\varepsilon\})} + \|u\|_{W_2^1(\{x\in\Omega:\, r_\pm^{(m)}<\varepsilon\delta_\varepsilon\})}\right),$$

where the constant C is independent of u and m. Summing up the last inequalities, we arrive at (3.2). □

We introduce an auxiliary function,

$$X = X(\xi,\eta) = \operatorname{Re}\ln(\sin z + \sqrt{\sin^2 z - \sin^2\eta}) - \xi_2, x$$
$$\xi = (\xi_1,\xi_2) = \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right), \qquad z = \xi_1 + \mathrm{i}\xi_2, \tag{3.5}$$

where the branches of the logarithm and the root are specified by the requirements $\ln 1 = 0$, $\sqrt{1} = 1$. We first define this function for $\xi_2 > 0$, $\xi_1 \neq \pi m$, $m \in \mathbb{Z}$, and then we extend it by the continuity to $\{\xi : \xi_2 \geqslant 0\}$. This function was introduced in [G1] and it was shown that it is harmonic as $\xi_2 > 0$, even and π-periodic w.r.t. ξ_1, decays exponentially as $\xi_2 \to +\infty$, and satisfies the boundary conditions

$$X = \ln\sin\eta \quad \text{on} \quad \gamma(\eta), \qquad \frac{\partial X}{\partial \xi_2} = -1 \quad \text{on} \quad \Gamma(\eta), \tag{3.6}$$

where

$$\gamma(\eta) := \{\xi : |\xi_1 - \pi m| < \eta,\ m\in\mathbb{Z},\ \xi_2 = 0\}, \qquad \Gamma(\eta) := O\xi_1\setminus\overline{\gamma(\eta)}. \tag{3.7}$$

The function X is continuous in $\{\xi : \xi_2 \geqslant 0\}$ and satisfies the estimate

$$|X| \leqslant |\ln\sin\eta| \tag{3.8}$$

uniformly in ξ. Indeed, since it is harmonic and decays exponentially as $\xi_2 \to +\infty$, it achieves its maximum on $O\xi_1$. Employing this fact and the explicit formula for X, one can easily check the estimate (3.8).

Lemma 3.2. *Given any $u \in \mathcal{D}(\mathcal{H}_\varepsilon)$, the function $uX\left(\frac{\cdot}{\varepsilon},\eta\right)$ belongs to $\mathring{W}_2^1(\Omega, \Gamma_+\cup\gamma_\varepsilon)$.*

Proof. The boundary conditions and the belongings $uX, X\nabla u \in L_2(\Omega)$ are due to the belonging $u \in \mathring{W}_2^1(\Omega,\Gamma_+\cup\gamma_\varepsilon)$ and the estimate (3.8). It remains to check that $u\nabla X \in L_2(\Omega)$. We employ the representation (3.1) for u and due to (3.2) we obtain $\overset{0}{u}\ \nabla X \in L_2(\Omega)$. To prove the belonging $\overset{1}{u}\ \nabla X \in L_2(\Omega)$, we integrate by parts taking into account the properties of X:

$$\begin{aligned}
\int_\Omega \nabla X\cdot|\overset{1}{u}|^2\nabla X\,\mathrm{d}x &= -\int_{\Gamma_-} X|\overset{1}{u}|^2\frac{\partial X}{\partial x_2}\,\mathrm{d}x_1 - \int_\Omega X\nabla X\cdot\nabla|\overset{1}{u}|^2\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} X|\overset{1}{u}|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \nabla X^2\cdot\nabla|\overset{1}{u}|^2\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} X|\overset{1}{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_-} X^2\frac{\partial|\overset{1}{u}|^2}{\partial x_2}\,\mathrm{d}x_1 + \frac{1}{2}\int_\Omega X^2\Delta|\overset{1}{u}|^2\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} X|\overset{1}{u}|^2\,\mathrm{d}x + \operatorname{Re}\int_{\Gamma_\varepsilon} X^2\overset{1}{u}\frac{\partial\overset{1}{u}}{\partial x_2}\,\mathrm{d}x_1 + \frac{1}{2}\int_\Omega X^2\Delta|\overset{1}{u}|^2\,\mathrm{d}x,
\end{aligned}$$

which by the estimate (3.8) and the belongings,

$$\overset{1}{u} \in W_2^2(\Omega), \qquad \overset{1}{u}, \frac{\partial \overset{1}{u}}{\partial x_2} \in L_2(\Gamma_\varepsilon),$$

implies $\overset{1}{u}\ \nabla X \in L_2(\Omega)$. □

Proof of theorem 2.1. Denote $u_\varepsilon := (\mathcal{H}_\varepsilon - \mathrm{i})^{-1} f$ for $f \in L_2(\Omega)$. By the definition of $\mathcal{H}_\varepsilon$, the function u_ε satisfies the identity,

$$(\nabla u_\varepsilon, \nabla\phi)_{L_2(\Omega)} + \mathrm{i}(u_\varepsilon, \phi)_{L_2(\Omega)} = (f, \phi)_{L_2(\Omega)}, \tag{3.9}$$

for any $\phi \in \mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$. For $\phi = u_\varepsilon$, we have

$$\|\nabla u_\varepsilon\|^2_{L_2(\Omega)} + \mathrm{i}\|u_\varepsilon\|^2_{L_2(\Omega)} = (f, u_\varepsilon)_{L_2(\Omega)}. \tag{3.10}$$

We take the imaginary part of the last identity and obtain

$$\begin{gathered}\|u_\varepsilon\|^2_{L_2(\Omega)} = \mathrm{Im}(f, u_\varepsilon)_{L_2(\Omega)} \leqslant \|f\|_{L_2(\Omega)}\|u_\varepsilon\|_{L_2(\Omega)},\\ \|u_\varepsilon\|_{L_2(\Omega)} \leqslant \|f\|_{L_2(\Omega)}.\end{gathered} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\|\nabla u_\varepsilon\|^2_{L_2(\Omega)} = \mathrm{Re}(f, u_\varepsilon)_{L_2(\Omega)} \leqslant \|f\|^2_{L_2(\Omega)}. \tag{3.12}$$

In the same way, for $u_0 := (\mathcal{H}_0 - \mathrm{i})^{-1} f$ we have the inequalities

$$\|u_0\|_{L_2(\Omega)} \leqslant \|f\|_{L_2(\Omega)}, \qquad \|\nabla u_0\|_{L_2(\Omega)} \leqslant \|f\|_{L_2(\Omega)}. \tag{3.13}$$

By lemma 3.2 the function $\phi = u_\varepsilon X$ belongs to $\mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$. We substitute it into (3.9):

$$(\nabla u_\varepsilon, X\nabla u_\varepsilon)_{L_2(\Omega)} + (\nabla u_\varepsilon, u_\varepsilon \nabla X)_{L_2(\Omega)} + \mathrm{i}(u_\varepsilon, X u_\varepsilon)_{L_2(\Omega)} = (f, X u_\varepsilon)_{L_2(\Omega)}. \tag{3.14}$$

We integrate by parts and employ the properties of X and (3.6):

$$\begin{aligned}\mathrm{Re}(\nabla u_\varepsilon, u_\varepsilon \nabla X)_{L_2(\Omega)} &= \frac{1}{2}\int_\Omega \nabla X \cdot (u_\varepsilon \nabla \overline{u}_\varepsilon + \overline{u}_\varepsilon \nabla u_\varepsilon)\,\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega \nabla X \cdot \nabla |u_\varepsilon|^2\,\mathrm{d}x = -\frac{1}{2}\int_{\Gamma_-} |u_\varepsilon|^2 \frac{\partial X}{\partial x_2}\,\mathrm{d}x_1 - \int_\Omega |u_\varepsilon|^2 \Delta X\,\mathrm{d}x\\ &= \frac{1}{2\varepsilon}\int_{\Gamma_\varepsilon} |u_\varepsilon|^2\,\mathrm{d}x_1.\end{aligned}$$

Now taking the real part of (3.14), we arrive at the identity,

$$(\nabla u_\varepsilon, X\nabla u_\varepsilon)_{L_2(\Omega)} + \frac{1}{2\varepsilon}\|u_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} = \mathrm{Re}(f, X u_\varepsilon)_{L_2(\Omega)}.$$

By (3.8), (3.11), (3.12) it yields

$$\begin{gathered}\frac{1}{2\varepsilon}\|u_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} \leqslant \mathrm{Re}(f, X u_\varepsilon)_{L_2(\Omega)} + |(\nabla u_\varepsilon, X\nabla u_\varepsilon)_{L_2(\Omega)}| \leqslant 2|\ln\sin\eta|\|f\|^2_{L_2(\Omega)},\\ \|u_\varepsilon\|_{L_2(\Gamma_-)} = \|u_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \leqslant 2\sqrt{\varepsilon|\ln\sin\eta(\varepsilon)|}\|f\|_{L_2(\Omega)}.\end{gathered} \tag{3.15}$$

Denote $v_\varepsilon := u_\varepsilon - u_0$. This function belongs to $\mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$ and is a generalized solution to the problem:

$$\begin{aligned}&-\Delta v_\varepsilon + \mathrm{i} v_\varepsilon = 0 \quad \text{in} \quad \Omega,\\ &v_\varepsilon = 0 \quad \text{on} \quad \Gamma_+ \cup \gamma_\varepsilon, \qquad \frac{\partial v_\varepsilon}{\partial x_2} = -\frac{\partial u_0}{\partial x_2} \quad \text{on} \quad \Gamma_\varepsilon.\end{aligned}$$

We multiply the equation by $\overline{v}_\varepsilon$ and integrate by parts:

$$-\int_{\Gamma_\varepsilon} \overline{v}_\varepsilon \frac{\partial u_0}{\partial x_2}\,dx_1 + \|\nabla v_\varepsilon\|^2_{L_2(\Omega)} + i\|v_\varepsilon\|^2_{L_2(\Omega)} = 0, \tag{3.16}$$

$$\begin{aligned}\|\nabla v_\varepsilon\|^2_{L_2(\Omega)} &= \operatorname{Re}\int_{\Gamma_\varepsilon} \overline{v}_\varepsilon \frac{\partial u_0}{\partial x_2}\,dx_1 = \operatorname{Re}\int_{\Gamma_\varepsilon} \overline{u}_\varepsilon \frac{\partial u_0}{\partial x_2}\,dx_1 \\ &\leqslant \|u_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \left\|\frac{\partial u_0}{\partial x_2}\right\|_{L_2(\Gamma_-)},\\ \|v_\varepsilon\|^2_{L_2(\Omega)} &\leqslant \operatorname{Im}\int_{\Gamma_\varepsilon} \overline{u}_\varepsilon \frac{\partial u_0}{\partial x_2}\,dx_1 \leqslant \|u_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \left\|\frac{\partial u_0}{\partial x_2}\right\|_{L_2(\Gamma_-)}.\end{aligned} \tag{3.17}$$

Let us estimate $\left\|\frac{\partial u_0}{\partial x_2}\right\|_{L_2(\Gamma_-)}$. For a.e. $x_1 \in \mathbb{R}$ we have

$$\frac{\partial u_0}{\partial x_2}(x_1, 0) = \frac{1}{\pi}\int_0^\pi \frac{\partial}{\partial x_2}(x_2 - \pi)\frac{\partial u_0}{\partial x_2}\,dx_2.$$

By the Cauchy–Schwarz inequality we derive

$$\begin{aligned}\left|\frac{\partial u_0}{\partial x_2}(x_1, 0)\right|^2 &\leqslant \frac{2}{\pi^2}\left(\int_0^\pi (x_2-\pi)^2\,dx_2 \int_0^\pi \left|\frac{\partial^2 u_0}{\partial x_2^2}(x)\right|^2 dx_2 + \int_0^\pi dx_2 \int_0^\pi \left|\frac{\partial u_0}{\partial x_2}(x)\right|^2 dx_2\right)\\ &= \frac{2}{\pi}\left(\frac{\pi^2}{3}\left\|\frac{\partial^2 u_0}{\partial x_2^2}(x_1, 0)\right\|^2_{L_2(0,\pi)} + \left\|\frac{\partial u_0}{\partial x_2}(x_1, 0)\right\|^2_{L_2(0,\pi)}\right).\end{aligned} \tag{3.18}$$

Proceeding as in the proof of lemma 7.1 in [LU, chapter 3, section 7], we check that

$$\left\|\frac{\partial^2 u_0}{\partial x_1^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u_0}{\partial x_2^2}\right\|^2_{L_2(\Omega)} + 2\left\|\frac{\partial^2 u_0}{\partial x_1 \partial x_2}\right\|^2_{L_2(\Omega)} = \|f + iu_0\|^2_{L_2(\Omega)},$$

and by (3.13) it implies

$$\left\|\frac{\partial^2 u_0}{\partial x_2^2}\right\|_{L_2(\Omega)} \leqslant \|f + iu_0\|_{L_2(\Omega)} \leqslant 2\|f\|_{L_2(\Omega)}.$$

This estimate and (3.18) yield

$$\left\|\frac{\partial u_0}{\partial x_2}\right\|^2_{L_2(\Gamma_-)} \leqslant \frac{2\pi}{3}\left\|\frac{\partial^2 u_0}{\partial x_2^2}\right\|^2_{L_2(\Omega)} + \frac{2}{\pi}\left\|\frac{\partial u_0}{\partial x_2}\right\|^2_{L_2(\Omega)} \leqslant \frac{8\pi^2 + 6}{3\pi}\|f\|^2_{L_2(\Omega)}.$$

Substituting this estimate and (3.15) into (3.17), we get

$$\|v_\varepsilon\|^2_{W_2^1(\Omega)} \leqslant 4\sqrt{\frac{8\pi^2+6}{3\pi}}\sqrt{\varepsilon|\ln\sin\eta(\varepsilon)|}\|f\|^2_{L_2(\Omega)} \leqslant \sqrt{13\varepsilon|\ln\sin\eta(\varepsilon)|}\|f\|^2_{L_2(\Omega)}$$

that completes the proof. □

Theorem 2.2 follows directly from theorem 2.1 and theorems VIII.23, VIII.24 in [RS1, chapter VIII, section 7].

4. Convergence of the resolvent of $\mathcal{H}_\varepsilon^{(p)}(\tau)$

In this section, we prove theorems 2.3 and 2.4. We begin with auxiliary lemmas.

Lemma 4.1. *The identity (2.3) holds true.*

Proof. Given $\lambda_n(\tau,\varepsilon)$, let $\psi_n(x,\tau,\varepsilon)$ be the associated eigenfunction. Employing the function $\mathrm{e}^{\frac{\mathrm{i}\tau x_1}{\varepsilon}}\psi_n(x,\tau,\varepsilon)$, one can construct easily a singular sequence for $\mathcal{H}_\varepsilon$ at $\lambda=\lambda_n(\tau,\varepsilon)$ and by the Weyl criterion we, therefore, obtain

$$\bigcup_{n=1}^{\infty}\{\lambda_n(\tau,\varepsilon):\tau\in[-1,1)\}\subseteq\sigma_{\mathrm{e}}(\mathcal{H}_\varepsilon).$$

Let

$$\lambda\notin\bigcup_{n=1}^{\infty}\{\lambda_n(\tau,\varepsilon):\tau\in[-1,1)\}. \tag{4.1}$$

It is sufficient to prove that $\lambda\notin\sigma(\mathcal{H}_\varepsilon)$. It is equivalent to the existence of the resolvent $(\mathcal{H}_\varepsilon-\lambda)^{-1}$. Let us prove the latter.

We introduce the Gelfand transformation $\mathcal{F}_\varepsilon$ w.r.t. x_1:

$$\begin{aligned}
&(\mathcal{F}_\varepsilon f)(x,\tau)=\mathrm{e}^{-\frac{\mathrm{i}\tau x_1}{\varepsilon}}(\mathcal{G}_\varepsilon f)(x,\tau),\qquad x\in\Omega_\varepsilon,\\
&(\mathcal{G}_\varepsilon f)(x,\tau):=\sum_{m\in\varepsilon\pi\mathbb{Z}}f(x_1-m,x_2)\,\mathrm{e}^{\frac{\mathrm{i}\tau m}{\varepsilon}},\qquad x\in\Omega_\varepsilon,\\
&\left(\mathcal{F}_\varepsilon^{-1}\widehat{f}\right)(x):=\frac{1}{2}\int_0^2\widehat{f}(x,\tau)\,\mathrm{e}^{\frac{\mathrm{i}\tau x_1}{\varepsilon}}\,\mathrm{d}\tau,\qquad x\in\Omega,
\end{aligned}$$

where it is assumed in the last formula that the functions defined on Ω_ε are extended $\varepsilon\pi$ periodically w.r.t. x_1.

Let X be a Hilbert space, and define

$$L_2((-1,1),X):=\int_{(-1,1)}^{\oplus}X.$$

Repeating the proof of theorem 2.2.5 in [K, chapter 2, section 2.2], one can prove easily that $\mathcal{G}_\varepsilon:L_2(\Omega)\to L_2((-1,1),L_2(\Omega_\varepsilon))$ is an isomorphism, and

$$\|\mathcal{G}_\varepsilon f\|^2_{L_2((-1,1),L_2(\Omega_\varepsilon))}=2\|f\|^2_{L_2(\Omega)},\qquad(\mathcal{G}_\varepsilon f,\mathcal{G}_\varepsilon g)_{L_2((-1,1),L_2(\Omega_\varepsilon))}=2(f,g)_{L_2(\Omega)}. \tag{4.2}$$

Similarly, $\mathcal{G}_\varepsilon:\mathring{W}_2^1(\Omega,\Gamma_+\cup\gamma_\varepsilon)\to L_2\big((-1,1),\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon)\big)$ is an isomorphism, and

$$\begin{aligned}
&\|\mathcal{G}_\varepsilon f\|^2_{L_2((-1,1),\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon))}=2\|f\|^2_{\mathring{W}_2^1(\Omega,\Gamma_+\cup\gamma_\varepsilon)},\\
&(\mathcal{G}_\varepsilon f,\mathcal{G}_\varepsilon g)_{L_2((-1,1),\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon))}=2(f,g)_{\mathring{W}_2^1(\Omega,\Gamma_+\cup\gamma_\varepsilon)}.
\end{aligned} \tag{4.3}$$

Given $f\in L_2(\Omega)$, let $\widehat{f}_\varepsilon(x,\tau):=(\mathcal{F}_\varepsilon f)(x,\tau)$. Due to (4.1), the operator $\big(\mathcal{H}_\varepsilon^{(p)}(\tau)-\lambda\big)^{-1}$ is invertible for each $\tau\in[-1,1)$. The function $\widehat{u}_\varepsilon=\widehat{u}_\varepsilon(x,\tau)$,

$$\widehat{u}_\varepsilon:=\big(\mathcal{H}_\varepsilon^{(p)}(\tau)-\lambda\big)^{-1}\widehat{f}\in\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon,\mathring{\Gamma}_+\cup\mathring{\gamma}_\varepsilon),$$

satisfies the uniform in τ estimate

$$\|\widehat{u}_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\leqslant C\|\widehat{f}\|_{L_2(\Omega_\varepsilon)}.$$

Hence, it belongs to $L_2((-1,1), \mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon, \mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon))$, and by (4.3) the function

$$u_\varepsilon(x) := \left(\mathcal{F}_\varepsilon^{-1}\widehat{u}_\varepsilon\right)(x) = \frac{1}{2}\int_0^2 \widehat{u}_\varepsilon(x,\tau)\,\mathrm{e}^{\frac{\mathrm{i}\tau x_1}{\varepsilon}}\,\mathrm{d}\tau$$

belongs to $\mathring{W}^1_2(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$.

Given any $\varphi \in \mathring{W}^1_2(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$, denote $\widehat{\varphi} := \mathcal{F}_\varepsilon\varphi$. The identities (4.2), (4.3) and the definition of $\widehat{u}_\varepsilon$ yield

$$\begin{aligned}\mathfrak{h}_\varepsilon[u,\varphi] - \lambda(u,\varphi)_{L_2(\Omega)} &= \frac{1}{2}\left(\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)\widehat{u}_\varepsilon, \left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)\widehat{\varphi}_\varepsilon\right)_{L_2((-1,1),L_2(\Omega_\varepsilon))}\\ &\quad + \frac{1}{2}\left(\frac{\partial \widehat{u}_\varepsilon}{\partial x_2}, \frac{\partial \widehat{\varphi}_\varepsilon}{\partial x_2}\right)_{L_2((-1,1),L_2(\Omega_\varepsilon))} - \frac{\lambda}{2}(\widehat{u}_\varepsilon, \widehat{\varphi}_\varepsilon)_{L_2((-1,1),L_2(\Omega_\varepsilon))}\\ &= \frac{1}{2}(\widehat{f}_\varepsilon, \widehat{\varphi}_\varepsilon)_{L_2((-1,1),L_2(\Omega_\varepsilon))} = (f,\varphi)_{L_2(\Omega)}.\end{aligned}$$

Thus, $u = (\mathcal{H}_\varepsilon - \lambda)^{-1} f$ and the operator $\mathcal{H}_\varepsilon - \lambda$ is boundedly invertible. □

Lemma 4.2. *Let $|\tau| < 1 - \delta$, where $0 < \delta < 1$ is a fixed constant, and*

$$u_\varepsilon = \left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} f.$$

Then

$$\|u_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant 4\|f\|_{L_2(\Omega_\varepsilon)}, \tag{4.4}$$

$$\left\|\frac{\partial u_\varepsilon}{\partial x_2}\right\|_{L_2(\Omega_\varepsilon)} \leqslant 2\|f\|_{L_2(\Omega_\varepsilon)}, \tag{4.5}$$

$$\left\|\frac{\partial u_\varepsilon}{\partial x_1}\right\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{2}{\delta^{1/2}}\|f\|_{L_2(\Omega_\varepsilon)}. \tag{4.6}$$

If, in addition, $f \in \mathfrak{L}^\perp$, then

$$\|u_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon}{\delta^{1/2}}\|f\|_{L_2(\Omega_\varepsilon)}, \qquad \|\nabla u_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon}{2\delta}\|f\|_{L_2(\Omega_\varepsilon)}. \tag{4.7}$$

Proof. Let us prove first that the resolvent $\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1}$ is well defined. The quadratic form corresponding to $\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}$ reads as follows:

$$\left(\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)u, u\right)_{L_2(\Omega_\varepsilon)} = \left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\|u\|^2_{L_2(\Omega_\varepsilon)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} \tag{4.8}$$

on $\mathring{W}^1_{2,\mathrm{per}}(\Omega_\varepsilon, \mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon)$. We can expand $u(\cdot, x_2)$ in terms of the basis $\left\{\mathrm{e}^{\pm\frac{2\mathrm{i}m x_1}{\varepsilon}}\right\}$, $m = 0, 1, 2, \ldots$. Employing this expansion, one can make sure that

$$\begin{aligned}\left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\|u\|^2_{L_2(\Omega_\varepsilon)} &= \left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u^\perp\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\|u^\perp\|^2_{L_2(\Omega_\varepsilon)}\\ &\geqslant \frac{4(1-|\tau|)}{\varepsilon^2}\|u^\perp\|^2_{L_2(\Omega_\varepsilon)} \geqslant \frac{4\delta}{\varepsilon^2}\|u^\perp\|^2_{L_2(\Omega_\varepsilon)},\end{aligned} \tag{4.9}$$

$$\left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\|u\|^2_{L_2(\Omega_\varepsilon)} \geqslant \delta\left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)}, \tag{4.10}$$

where $u^\perp$ is the projection of u on $\mathfrak{L}^\perp$. It follows from (3.4) that

$$\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} \geqslant \frac{1}{4}\|u\|^2_{L_2(\Omega_\varepsilon)} \tag{4.11}$$

for $u \in \mathring{W}_2^1(\Omega_\varepsilon, \mathring{\Gamma}_+)$. The estimates (4.11) and (4.9) imply that

$$\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2} \geqslant \frac{1}{4},$$

and therefore the inverse of this operator is well defined and satisfies the estimate (4.4). In view of (4.8), we thus have

$$\left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u_\varepsilon\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\|u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \left\|\frac{\partial u_\varepsilon}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} = (f, u_\varepsilon)_{L_2(\Omega_\varepsilon)}. \tag{4.12}$$

This identity, (4.4) and (4.10) imply

$$\left\|\frac{\partial u_\varepsilon}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} \leqslant 4\|f\|^2_{L_2(\Omega_\varepsilon)}, \qquad \delta\left\|\frac{\partial u_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)} \leqslant 4\|f\|^2_{L_2(\Omega_\varepsilon)}$$

that proves (4.5) and (4.6).

Assume that $f \in \mathfrak{L}^\perp$ and let $u_\varepsilon^\perp$ be the projection of u_ε on $\mathfrak{L}^\perp$. Then it follows from (4.12) and (4.9) that

$$\left\|\left(\mathrm{i}\frac{\partial}{\partial x_1} - \frac{\tau}{\varepsilon}\right)u_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)} - \frac{\tau^2}{\varepsilon^2}\left\|u_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)} + \left\|\frac{\partial u_\varepsilon}{\partial x_2}\right\|^2_{L_2(\Omega_\varepsilon)} = \left(f, u_\varepsilon^\perp\right)_{L_2(\Omega_\varepsilon)}.$$

We substitute the estimate (4.9) into the last identity:

$$\frac{4\delta}{\varepsilon^2}\left\|u_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)} \leqslant \|f\|_{L_2(\Omega_\varepsilon)}\left\|u_\varepsilon^\perp\right\|_{L_2(\Omega_\varepsilon)},$$

$$\left\|u_\varepsilon^\perp\right\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon^2}{4\delta}\|f\|_{L_2(\Omega_\varepsilon)},$$

$$|(f, u_\varepsilon)_{L_2(\Omega_\varepsilon)}| = \left|\left(f, u_\varepsilon^\perp\right)_{L_2(\Omega_\varepsilon)}\right| \leqslant \frac{\varepsilon^2}{4\delta}\|f\|^2_{L_2(\Omega_\varepsilon)}.$$

The last estimate, (4.10), (4.11), (4.12) yield

$$\|u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon^2}{\delta}\|f\|^2_{L_2(\Omega_\varepsilon)}, \qquad \|\nabla u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon^2}{4\delta^2}\|f\|^2_{L_2(\Omega_\varepsilon)},$$

that completes the proof. □

Lemma 4.3. *Let $F = F(x_2) \in L_2(0, \pi)$ and $U := \mathcal{Q}^{-1}F$. Then*

$$|U'(0)| \leqslant \sqrt{\frac{\pi}{3}}\|F\|_{L_2(0,\pi)}.$$

Proof. It is easy to find the function U explicitly:

$$U(x_2) = -\frac{1}{2}\int_0^\pi \left(|x_2 - t| - x_2 - t + \frac{2x_2 t}{\pi}\right) F(t)\,\mathrm{d}t.$$

Hence,

$$U'(0) = \int_0^\pi \left(1 - \frac{t}{\pi}\right) F(t)\,\mathrm{d}t,$$

and by the Cauchy–Schwarz inequality, we obtain

$$|U'(0)| \leqslant \left(\int_0^\pi \left(1-\frac{t}{\pi}\right)^2 \mathrm{d}t\right)^{1/2} \|F\|_{L_2(0,\pi)} = \sqrt{\frac{\pi}{3}}\|F\|_{L_2(0,\pi)}. \qquad \square$$

Lemma 4.4. *Each function* $u \in \mathcal{D}\big(\mathcal{H}_\varepsilon^{(p)}(\tau)\big)$ *can be represented as*

$$\begin{aligned} u(x) &= \overset{0}{u}(x) + \overset{1}{u}(x),\\ \overset{0}{u}(x) &= \alpha_-\chi\left(\frac{3r_-^{(0)}}{\varepsilon\delta_\varepsilon}\right)\sqrt{r_-^{(0)}}\sin\frac{\theta_-^{(0)}}{2} + \alpha_+\chi\left(\frac{3r_+^{(0)}}{\varepsilon\delta_\varepsilon}\right)\sqrt{r_+^{(0)}}\sin\frac{\theta_+^{(0)}}{2}, \end{aligned} \tag{4.13}$$

where $\overset{1}{u}(x) \in W_2^2(\Omega_\varepsilon)$ *vanishes on* $\mathring{G}_+ \cup \mathring{\gamma}_\varepsilon$ *and satisfies the periodic boundary condition on the lateral boundaries of* Ω_ε. *Here* $\alpha_\pm$ *are some constants, and* δ_ε *is the same as in lemma 3.1.*

The proof is completely analogous to that of lemma 3.1.

Proof of theorem 2.3. Given $f \in L_2(\Omega_\varepsilon)$, we decompose it as $f = F_\varepsilon + f_\varepsilon^\perp$, $F_\varepsilon \in \mathfrak{L}$, $f_\varepsilon^\perp \in \mathfrak{L}^\perp$,

$$\|F_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \left\|f_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)} = \|f\|^2_{L_2(\Omega_\varepsilon)}. \tag{4.14}$$

Then

$$\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} f = \left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} F_\varepsilon + \left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} f_\varepsilon^\perp,$$

and by (4.7), (4.8), (4.14) we obtain immediately

$$\left\|\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} f_\varepsilon^\perp\right\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon}{\delta^{1/2}}\left\|f_\varepsilon^\perp\right\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon}{\delta^{1/2}}\|f\|_{L_2(\Omega_\varepsilon)}. \tag{4.15}$$

It remains to construct an appropriate approximation for

$$u_\varepsilon := \left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} F_\varepsilon.$$

It is clear that $(\mathcal{Q}^{-1} \oplus 0)f = \mathcal{Q}^{-1}F_\varepsilon$. We denote this function by U_ε. Let χ be a cut-off function defined before lemma 3.1. We introduce one more function:

$$\widehat{u}_\varepsilon(x) := U_\varepsilon(x_2) + \varepsilon U_\varepsilon'(0)\left(X\left(\frac{x}{\varepsilon}, \eta(\varepsilon)\right) - \ln\sin\eta(\varepsilon)\right)\chi(x_2).$$

It is straightforward to check that $\widehat{u}_\varepsilon$ satisfies the periodic boundary condition on the lateral surfaces of Ω_ε, vanishes on $\mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon$, and obeys the Neumann condition on $\mathring{\Gamma}_\varepsilon$. It also belongs to the domain of the operator $\mathcal{H}_\varepsilon^{(p)}(\tau)$ since the function $\chi(x_2)X\left(\frac{x}{\varepsilon}, \eta(\varepsilon)\right)$ satisfies the representation (4.13).

Employing the properties of X, we see that

$$\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\widehat{u}_\varepsilon = F_\varepsilon - U_\varepsilon'(0)\left(2\mathrm{i}\tau\chi\frac{\partial X}{\partial x_1} + \varepsilon\chi''(X - \ln\sin\eta) + 2\varepsilon\chi'\frac{\partial X}{\partial x_2}\right),$$

and for $\widetilde{u}_\varepsilon := u_\varepsilon - \widehat{u}_\varepsilon$ we have

$$\begin{aligned} \widetilde{u}_\varepsilon &= U_\varepsilon'(0)\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} g_\varepsilon - \varepsilon U_\varepsilon'(0)\ln\sin\eta\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1}\chi''\\ &= \widetilde{u}_\varepsilon^{(1)} + \widetilde{u}_\varepsilon^{(2)},\\ g_\varepsilon &= 2\mathrm{i}\tau\chi\frac{\partial X}{\partial x_1} + \varepsilon\left(\chi''X + 2\chi'\frac{\partial X}{\partial x_2}\right). \end{aligned}$$

It follows from [Bo5, lemma 3.7] that

$$\int_{-\pi/2}^{\pi/2} X(\xi,\eta)\,\mathrm{d}\xi_1 = 0 \qquad \text{for} \quad \xi_2 > 0.$$

Hence,

$$\int_{-\varepsilon\pi/2}^{\varepsilon\pi/2} X\left(\frac{x}{\varepsilon},\eta(\varepsilon)\right)\mathrm{d}x_1 = 0 \qquad \text{for} \quad 0 < x_2 < \pi,$$

and $g \in \mathfrak{L}^{\perp}$. By (4.7) it implies that

$$\|\widetilde{u}_\varepsilon^{(1)}\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{\varepsilon}{\delta^{1/2}}|U_\varepsilon'(0)|\|g\|_{L_2(\Omega_\varepsilon)}. \tag{4.16}$$

The identity

$$\|\nabla_\xi X\|^2_{L_2(\Pi)} = \pi|\ln\sin\eta|, \qquad \Pi := \{\xi : |\xi_1| < \pi/2, \xi_2 > 0\},$$

was proven in [Bo5, lemma 3.8]. Together with (3.8) and (2.1) it yields

$$\begin{aligned}\|g\|_{L_2(\Omega_\varepsilon)} &\leqslant 2\left\|\frac{\partial X}{\partial x_1}\right\|_{L_2(\Omega_\varepsilon)} + \varepsilon|\ln\sin\eta|\|\chi''\|_{L_2(\Omega_\varepsilon)} + 2\varepsilon C\left\|\frac{\partial X}{\partial x_2}\right\|_{L_2(\Omega_\varepsilon)}\\ &\leqslant 2\sqrt{2}\|\nabla_x X\|_{L_2(\Omega_\varepsilon)} + C\varepsilon^{3/2}|\ln\sin\eta|\\ &\leqslant 2\sqrt{2}\|\nabla_\xi X\|_{L_2(\Pi)} + C\varepsilon^{3/2}|\ln\sin\eta|\\ &= 2\sqrt{2\pi}|\ln\sin\eta|^{1/2} + C\varepsilon^{3/2}|\ln\sin\eta| \leqslant 6|\ln\sin\eta|^{1/2},\end{aligned} \tag{4.17}$$

if ε is small enough. Here the symbol C indicates inessential constants independent of ε and η. Since

$$F_\varepsilon(x_2) = (\varepsilon\pi)^{-1}\int_{-\frac{\varepsilon\pi}{2}}^{\frac{\varepsilon\pi}{2}} f(x)\,\mathrm{d}x_1,$$

by the Cauchy–Schwarz inequality we have

$$\|F_\varepsilon\|_{L_2(0,\pi)} \leqslant (\varepsilon\pi)^{-1/2}\|f\|_{L_2(\Omega_\varepsilon)}.$$

This estimate and lemma 4.3 yield

$$|U_\varepsilon'(0)| \leqslant \frac{\varepsilon^{-1/2}}{\sqrt{3}}\|f\|_{L_2(\Omega_\varepsilon)}. \tag{4.18}$$

It follows from (4.16) and (4.17), and the last estimate that

$$\|\widetilde{u}_\varepsilon^{(1)}\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{2\sqrt{3}}{\delta^{1/2}}\varepsilon^{1/2}|\ln\sin\eta|^{1/2}\|f\|_{L_2(\Omega_\varepsilon)}.$$

Since $\|\chi''\|_{L_2(\Omega_\varepsilon)} = \sqrt{\varepsilon\pi}\|\chi''\|_{L_2(0,\pi)}$, by (4.4) and (4.18) we derive

$$\|\widetilde{u}_\varepsilon^{(2)}\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon|\ln\sin\eta|\|f\|_{L_2(\Omega_\varepsilon)},$$

where C is a constant independent of ε and η. Thus,

$$\|\widetilde{u}_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{4\varepsilon^{1/2}}{\delta^{1/2}}|\ln\sin\eta|^{1/2}\|f\|_{L_2(\Omega_\varepsilon)}, \tag{4.19}$$

if ε is small enough. It follows from (4.18) and (3.8) that

$$\|\varepsilon U_\varepsilon'(0)(X - \ln\sin\eta)\chi\|_{L_2(\Omega_\varepsilon)} \leqslant \frac{2\varepsilon\pi}{\sqrt{3}}|\ln\sin\eta|\|f\|_{L_2(\Omega_\varepsilon)}.$$

Hence, by (4.15), (4.19), (2.1) we obtain

$$\left\| \left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)^{-1} f - (\mathcal{Q}^{-1} \oplus 0) f \right\|_{L_2(\Omega_\varepsilon)}$$
$$\leqslant \left(\frac{\varepsilon}{\delta^{1/2}} + \frac{4}{\delta^{1/2}}\varepsilon^{1/2}|\ln\sin\eta|^{1/2} + \frac{2\pi}{\sqrt{3}}\varepsilon|\ln\sin\eta|\right)\|f\|_{L_2(\Omega_\varepsilon)}$$
$$\leqslant \frac{\varepsilon + 5\varepsilon^{1/2}|\ln\sin\eta|^{1/2}}{\delta^{1/2}}\|f\|_{L_2(\Omega_\varepsilon)},$$

if ε is small enough. □

Proof of theorem 2.4. By the standard bracketing arguments (see, for instance, [RS2, chapter XIII, section 15, proposition 4]) we see that the eigenvalues of $\mathcal{H}_\varepsilon^{(p)} - \frac{\tau^2}{\varepsilon^2}$ are estimated from above by those of the same operator in the case $\eta = \pi/2$. In other words, we increase the eigenvalues of $\mathcal{H}_\varepsilon^{(p)} - \frac{\tau^2}{\varepsilon^2}$, if we replace the Neumann condition on $\mathring{\Gamma}_\varepsilon$ by the Dirichlet one. In the latter case, given any N there exists $\varepsilon_0 > 0$ so that for $\varepsilon < \varepsilon_0$ the first N eigenvalues are n^2 with the eigenfunctions $\sin nx_2$. Hence,

$$0 \leqslant \lambda_n(\tau,\varepsilon) - \frac{\tau^2}{\varepsilon^2} \leqslant n^2, \qquad n \leqslant N, \qquad \varepsilon < \varepsilon_0. \tag{4.20}$$

By [OSI, chapter III, section 1, theorem 1.4] and by theorem 2.3 we have

$$\left|\frac{1}{\lambda_n(\tau,\varepsilon) - \frac{\tau^2}{\varepsilon^2}} - \frac{1}{n^2}\right| \leqslant \frac{\varepsilon + 5\varepsilon^{1/2}|\ln\sin\eta|^{1/2}}{\delta^{1/2}}.$$

The statement of the theorem follows from the two previous estimates. □

5. Bottom of the spectrum

In this section, we prove theorem 2.5. First we prove that the eigenvalue $\lambda_1(\tau,\varepsilon)$ attains its minimum at $\tau = 0$.

In the same way as in the proof of theorem 2.4, by the bracketing arguments we see that the eigenvalues of $\mathcal{H}_\varepsilon^{(p)}(\tau)$ are estimated from below by those of the same operator with $\eta = 0$, i.e., when we replace the Dirichlet condition on $\mathring{\Gamma}_\varepsilon$ by the Neumann one. The lowest eigenvalue of the latter operator is $\frac{1}{4} + \frac{\tau^2}{\varepsilon^2}$ and therefore

$$\begin{aligned} &\lambda_1(\tau,\varepsilon) \geqslant \frac{1}{4} + \frac{\tau^2}{\varepsilon^2}, \qquad \tau \in [-1,1), \\ &\lambda_1(\tau,\varepsilon) \geqslant \frac{5}{4}, \qquad |\tau| \geqslant \varepsilon. \end{aligned} \tag{5.1}$$

Since by (2.4) the eigenvalue $\lambda_1(0,\varepsilon)$ behaves as

$$\lambda_1(0,\varepsilon) = 1 + o(1), \qquad \varepsilon \to +0, \tag{5.2}$$

in view of (5.1) we conclude that $\lambda_1(\tau,\varepsilon) \geqslant \lambda_1(0,\varepsilon)$ as $|\tau| \geqslant \varepsilon$ for sufficiently small ε, and thus

$$\inf_{\tau\in[-1,1)} \lambda_1(\tau,\varepsilon) = \inf_{\tau\in[-\varepsilon,\varepsilon]} \lambda_1(\tau,\varepsilon).$$

Consider the case $|\tau| \leqslant \varepsilon$. For such τ, the eigenvalue $\lambda_1(\tau,\varepsilon)$ is simple as it follows from (2.4). Let $\psi_\varepsilon = \psi_\varepsilon(x)$ be the real-valued eigenfunction associated with $\lambda_1(0,\varepsilon)$ normalized in $L_2(\Omega_\varepsilon)$.

Lemma 5.1. *The convergence*

$$\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|_{L_2(\Omega_\varepsilon)} \to 0, \qquad \varepsilon\to+0,$$

holds true.

Proof. By the definition, the function ψ_ε satisfies the identity,

$$\|\nabla\psi_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = \lambda_1(0,\varepsilon). \tag{5.3}$$

Let $\psi_\varepsilon^\perp$ be the projection of ψ_ε on $\mathfrak{L}^\perp$, and $\mathring{\psi}_\varepsilon := \psi_\varepsilon - \psi_\varepsilon^\perp \in \mathfrak{L}$, $\mathring{\psi}_\varepsilon = \mathring{\psi}_\varepsilon(x_2)$. By the inequality (4.9) with $\tau = 0$, we obtain

$$\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)} \geqslant 4\varepsilon^{-2}\left\|\psi_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)}.$$

Together with (5.2) and (5.3) it yields

$$\left\|\psi_\varepsilon^\perp\right\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon), \qquad \varepsilon\to+0.$$

Since

$$\|\mathring{\psi}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \|\psi^\perp\|^2_{L_2(\Omega_\varepsilon)} = \|\psi_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = 1,$$

it follows that

$$\|\mathring{\psi}_\varepsilon\|_{L_2(\Omega_\varepsilon)} = 1 + \mathcal{O}(\varepsilon), \qquad \varepsilon\to+0. \tag{5.4}$$

We integrate the equation

$$-\Delta\psi_\varepsilon = \lambda_1(0,\varepsilon)\psi_\varepsilon$$

w.r.t. $x_1 \in (-\varepsilon\pi/2, \varepsilon\pi/2)$ for $x_2 \in (0,\pi)$:

$$-\frac{\mathrm{d}^2\mathring{\psi}_\varepsilon}{\mathrm{d}x_2^2} = \lambda_1(0,\varepsilon)\mathring{\psi}_\varepsilon, \qquad x_2\in(0,\pi),\ \mathring{\psi}_\varepsilon(\pi) = 0.$$

Hence,

$$\mathring{\psi}_\varepsilon(x_2) = C_\varepsilon(\varepsilon\pi)^{-1/2}\sin\sqrt{\lambda_1(0,\varepsilon)}(\pi - x_2),$$

$$\|\mathring{\psi}_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = \frac{C_\varepsilon^2}{2}\left(\pi - \frac{\sin 2\sqrt{\lambda_1(0,\varepsilon)}\pi}{2\sqrt{\lambda_1(0,\varepsilon)}}\right),$$

where C_ε is a constant. If follows from (5.4) and (5.2) that

$$C_\varepsilon^2 = \frac{2}{\pi} + o(1), \qquad \varepsilon\to+0.$$

By direct calculations we check that

$$\left\|\frac{\mathrm{d}\mathring{\psi}_\varepsilon}{\mathrm{d}x_2}\right\|^2_{L_2(\Omega_\varepsilon)} = \frac{C_\varepsilon^2\lambda_1(0,\varepsilon)}{2}\left(\pi + \frac{\sin 2\sqrt{\lambda_1(0,\varepsilon)}\pi}{2\sqrt{\lambda_1(0,\varepsilon)}}\right) = \lambda_1(0,\varepsilon)(1+o(1)), \quad \varepsilon\to+0.$$

We substitute this identity and the asymptotics (5.2) into (5.3):

$$\left\|\nabla\psi_\varepsilon^\perp\right\|^2_{L_2(\Omega_\varepsilon)} = o(1), \qquad \left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)} = \left\|\frac{\partial\psi_\varepsilon^\perp}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)} = o(1), \quad \varepsilon\to+0.$$

□

Applying the bracketing arguments in the same way as above, we can estimate the eigenvalues $\lambda_1(\tau,\varepsilon)$ and $\lambda_2(\tau,\varepsilon)$ as

$$\frac{1}{4} \leqslant \lambda_1(\tau,\varepsilon) - \frac{\tau^2}{\varepsilon^2} \leqslant 1, \qquad \frac{9}{4} \leqslant \lambda_2(\tau,\varepsilon) - \frac{\tau^2}{\varepsilon^2} \leqslant 4,$$

for ε small enough, $|\tau| \leqslant \varepsilon$. One can make sure easily that

$$\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon = \lambda_1(0,\varepsilon)\psi_\varepsilon - \frac{2\mathrm{i}\tau}{\varepsilon}\frac{\partial\psi_\varepsilon}{\partial x_1},$$
$$\left(\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon, \psi_\varepsilon\right)_{L_2(\Omega_\varepsilon)} = \lambda_1(0,\varepsilon),$$
$$\left\|\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon\right\|^2_{L_2(\Omega_\varepsilon)} = \lambda_1^2(0,\varepsilon) + \frac{4\tau^2}{\varepsilon^2}\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)}.$$

Employing these formulae, we apply the Temple inequality (see [D, chapter 4, section 4.6, theorem 4.6.3]) to the operator $\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}$:

$$\lambda_1(\tau,\varepsilon) - \frac{\tau^2}{\varepsilon^2} \geqslant \frac{\frac{9}{4}\left(\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon, \psi_\varepsilon\right)_{L_2(\Omega_\varepsilon)} - \left\|\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon\right\|^2_{L_2(\Omega_\varepsilon)}}{\frac{9}{4} - \left(\left(\mathcal{H}_\varepsilon^{(p)}(\tau) - \frac{\tau^2}{\varepsilon^2}\right)\psi_\varepsilon, \psi_\varepsilon\right)_{L_2(\Omega_\varepsilon)}}$$
$$= \frac{\frac{9}{4}\lambda_1(0,\varepsilon) - \lambda_1(0,\varepsilon)^2 - \frac{4\tau^2}{\varepsilon^2}\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)}}{\frac{9}{4} - \lambda_1(0,\varepsilon)}$$
$$= \lambda_1(0,\varepsilon) - \frac{4\tau^2}{\varepsilon^2\left(\frac{9}{4} - \lambda_1(0,\varepsilon)\right)}\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)}.$$

Hence, by lemma 5.1 and the asymptotics (5.2)

$$\lambda_1(\tau,\varepsilon) \geqslant \lambda_1(0,\varepsilon) + \frac{\tau^2}{\varepsilon^2}\left(1 - \frac{16}{9 - 4\lambda_1(0,\varepsilon)}\left\|\frac{\partial\psi_\varepsilon}{\partial x_1}\right\|^2_{L_2(\Omega_\varepsilon)}\right) \geqslant \lambda_1(0,\varepsilon), \qquad \tau \in [-\varepsilon,\varepsilon],$$

if ε is small enough. It proves the identity (2.5).

We proceed to the asymptotics for $\lambda_1(0,\varepsilon)$. We construct it first formally and then we justify it. The formal constructing is based on the boundary layer method, and in fact it follows the main ideas of [G1].

We construct the asymptotics for $\lambda_1(0,\varepsilon)$ as the series (2.6). The asymptotics for the associated eigenfunction is constructed as

$$\widetilde{\psi}_\varepsilon(x) = \Psi_\varepsilon^{\mathrm{in}}(x,\eta) + \chi(x_2)\Psi_\varepsilon^{\mathrm{bl}}(\xi,\eta),$$
$$\Psi_\varepsilon^{\mathrm{in}}(x,\eta) = \sin\sqrt{\lambda_1(0,\varepsilon)}(\pi - x_2),$$

where the cut-off function χ was defined before lemma 3.1, and the variables ξ were introduced in (3.5). In contrast to ψ_ε, the function $\widetilde{\psi}_\varepsilon$ is not supposed to be normalized in $L_2(\Omega_\varepsilon)$.

The function $\Psi_\varepsilon^{\mathrm{bl}}$ is a boundary layer at $\mathring{\Gamma}_-$ and its asymptotics is sought as

$$\Psi_\varepsilon^{\mathrm{bl}}(\xi,\eta) = \sum_{i=1}^{\infty}\varepsilon^i v_i(\xi,\eta). \tag{5.5}$$

The main aim of the formal constructing is to determine the numbers μ_i and the functions v_i.

It is clear that

$$\Psi_\varepsilon^{\mathrm{in}}(0,\eta) = \sin\sqrt{\lambda_1(0,\varepsilon)}\pi = \sum_{i=1}^{\infty}\varepsilon^i\left(-\frac{\pi}{2}\mu_i + G_i^{(\mathrm{D})}(\mu_1,\ldots,\mu_{i-1})\right),$$
$$\frac{\mathrm{d}\Psi_\varepsilon^{\mathrm{in}}}{\mathrm{d}x_2}(0,\eta) = -\sqrt{\lambda_1(0,\varepsilon)}\cos\sqrt{\lambda_1(0,\varepsilon)}\pi \tag{5.6}$$
$$= 1 + \sum_{i=1}^{\infty}\varepsilon^i\left(\frac{\mu_i}{2} + G_i^{(\mathrm{N})}(\mu_1,\ldots,\mu_{i-1})\right),$$

where $G_i^{(D)}$, $G_i^{(N)}$ are some polynomials, and, in particular,

$$G_1^{(D)} = 0, \qquad G_1^{(N)} = 0, \qquad G_2^{(D)}(\mu_1) = \frac{\pi}{8}\mu_1^2. \tag{5.7}$$

The function $\widetilde{\psi}_\varepsilon$ satisfies the boundary condition on $\mathring{\gamma}_\varepsilon$ and $\mathring{\Gamma}_\varepsilon$, and by (5.5) and (5.6) it implies the boundary conditions for v_i:

$$\frac{\partial v_1}{\partial \xi_2} = -1, \qquad \frac{\partial v_i}{\partial \xi_2} = -\frac{1}{2}\mu_{i-1} - G_{i-1}^{(N)}(\mu_1, \dots, \mu_{i-2}), \qquad \xi \in \mathring{\Gamma}(\eta), \quad i \geqslant 2, \tag{5.8}$$

$$\begin{aligned} &v_i = \frac{\pi}{2}\mu_i - G_i^{(D)}(\mu_1, \dots, \mu_{i-1}), \qquad \xi \in \mathring{\gamma}(\eta), \qquad i \geqslant 1, \\ &\mathring{\gamma}_\eta := \partial\Pi \cap \gamma(\eta), \qquad \mathring{\Gamma}(\eta) := \partial\Pi \cap \Gamma(\eta), \end{aligned} \tag{5.9}$$

where, we recall, the sets $\gamma(\eta)$ and $\Gamma(\eta)$ were introduced in (3.7). The functions v_i should satisfy the periodic boundary conditions on the lateral boundaries of Π, since the same is assumed for ψ_ε. And they should decay exponentially as $\xi_2 \to +\infty$, since they are boundary layer functions.

In order to obtain the equations for v_i, we substitute the series (2.6), (5.5) into the equation

$$-\Delta\Psi_\varepsilon^{\mathrm{bl}} = \lambda_1(0, \varepsilon)\Psi_\varepsilon^{\mathrm{bl}}, \qquad x \in \Omega_\varepsilon,$$

pass to the variables ξ, and equate the coefficients of the same powers of ε. It implies

$$-\Delta_\xi v_i = \sum_{j=0}^{i-3} \mu_j v_{i-j-2}, \quad \xi \in \Pi, \tag{5.10}$$

where $\mu_0 := 1$. The functions v_1, v_2 are harmonic ones and we can find them explicitly:

$$v_1 = X, \qquad v_2 = \frac{\mu_1}{2}X,$$

where, we recall, the function X was introduced in (3.5). It follows from (3.6) that

$$v_1 = \ln\sin\eta, \qquad v_2 = \frac{\mu_1}{2}\ln\sin\eta, \qquad \xi \in \mathring{\gamma}(\eta),$$

and by (5.7)–(5.9) we obtain

$$\frac{\pi}{2}\mu_1 = \ln\sin\eta, \qquad \frac{\mu_1}{2}\ln\sin\eta = \frac{\pi}{2}\mu_2 - \frac{\pi}{8}\mu_1^2.$$

The identity obtained leads us directly to formulae (2.7).

The solvability condition of the problems (5.10), (5.8), (5.9) for $i \geqslant 3$ is given by lemma 3.1 in [G1]:

$$\begin{aligned} &\pi\left(\frac{\pi}{2}\mu_i - G_i^{(D)} - \left(\frac{1}{2}\mu_{i-1} + G_{i-1}^{(N)}\right)\ln\sin\eta\right) = \sum_{j=0}^{i-3}\mu_j \int_\Pi Y v_{i-j-2}\,\mathrm{d}\xi, \\ &Y = Y(\xi, \eta) := X(\xi, \eta) + \xi_2 - \ln\sin\eta. \end{aligned}$$

It implies the formulae for μ_i:

$$\begin{aligned} \mu_i = \frac{2}{\pi}\Bigg(&\frac{1}{\pi}\sum_{j=0}^{i-3}\mu_j\int_\Pi Y v_{i-j-2}\,\mathrm{d}\xi + G_i^{(D)}(\mu_1, \dots, \mu_{i-1}) \\ &+ \left(\frac{\mu_{i-1}}{2} + G_{i-1}^{(N)}(\mu_1, \dots, \mu_{i-2})\right)\ln\sin\eta\Bigg). \end{aligned} \tag{5.11}$$

So, the problems (5.10), (5.8), (5.9) are solvable.

By induction it follows from [Bo5, lemma 3.7] that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_i(\xi,\eta)\,\mathrm{d}\xi_1 = 0 \qquad \text{for all} \qquad \xi_2 > 0.$$

It allows us to apply theorem 3.1 from [Bo5] to the problems (5.10), (5.8), (5.9). Similar in fact was the core of the proof of lemma 4.2 in [Bo5], and repeating word by word the proof of this lemma, we arrive at

Lemma 5.2. *As $\eta \to +0$, the identities (2.8) and the uniform in η estimates*

$$\|\xi_2^p v_j\|_{L_2(\Pi)} \leqslant C|\ln\eta|^{j-1}, \qquad \left\|\xi_2^{p+1}\nabla_\xi \frac{\partial v_j}{\partial \xi_2}\right\|_{L_2(\Pi)} \leqslant C|\ln\eta|^{j-1}$$

$$\left\|\xi_2^m \nabla_\xi \frac{\partial^m v_j}{\partial \xi_2^m}\right\|_{L_2(\Pi)} \leqslant C|\ln\eta|^{j-\frac{1}{2}}, \qquad m = 0, 1,$$

hold true, where $p \geqslant 0$.

Employing this lemma and reproducing the proof of lemma 5.1 in [Bo5], one can prove easily

Lemma 5.3. *For any $p \geqslant 2$, $R > 1$ the uniform in R and η estimates*

$$\|v_j\|_{L_2(\Pi_R)} \leqslant CR^{-p+1}(|\ln\eta|^{j-1}+1), \qquad \|\nabla_\xi v_j\|_{L_2(\Pi_R)} \leqslant CR^{-p+1}(|\ln\eta|^{j-1}+1),$$

hold true.

Given $M \geqslant 2$, denote

$$\Lambda_\varepsilon^{(M)} := 1 + \sum_{i=1}^{M} \varepsilon^i \mu_i(\eta), \qquad \widetilde{\Psi}_\varepsilon^{(M)}(x) := \sin\sqrt{\Lambda_\varepsilon^{(M)}}(\pi - x_2) + \chi(x_2)\sum_{i=1}^{M}\varepsilon^i v_i(\xi,\eta).$$

Lemma 5.4. *The function $\widetilde{\Psi}_\varepsilon^{(M)} \in W_2^1(\Omega_\varepsilon)$ satisfies the periodic boundary condition on the lateral boundaries of Ω_ε and is a generalized solution to the problem:*

$$-\Delta\widetilde{\Psi}_\varepsilon^{(M)} = \Lambda_\varepsilon^{(M)}\widetilde{\Psi}_\varepsilon^{(M)} + \widetilde{f}_\varepsilon^{(M)} \quad in \quad \Omega_\varepsilon, \qquad \widetilde{\Psi}_\varepsilon^{(M)} = 0 \quad on \quad \mathring{\Gamma}_+,$$

$$\widetilde{\Psi}_\varepsilon^{(M)} = B_{\varepsilon,\mathrm{D}}^{(M)} \quad on \quad \mathring{\gamma}_\varepsilon, \ \frac{\partial\widetilde{\Psi}_\varepsilon^{(M)}}{\partial x_2} = B_{\varepsilon,\mathrm{N}}^{(M)} \quad on \quad \mathring{\Gamma}_\varepsilon,$$

where $\widetilde{f}_\varepsilon^{(M)} \in L_2(\Omega_\varepsilon)$, $B_{\varepsilon,\mathrm{D}}^{(M)}$, $B_{\varepsilon,\mathrm{N}}^{(M)}$ are constants. The uniform in ε and η estimates

$$\left\|\widetilde{f}_\varepsilon^{(M)}\right\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^M(|\ln\eta|^{M-2}+1),$$

$$\left|B_{\varepsilon,\mathrm{N}}^{(M)}\right| \leqslant C\varepsilon^M(|\ln\eta|^M+1), \qquad \left|B_{\varepsilon,\mathrm{D}}^{(M)}\right| \leqslant C\varepsilon^{M+1}(|\ln\eta|^{M+1}+1),$$

hold true.

This lemma can be checked by direct calculations with employing lemma 5.3.

We let

$$\Psi_\varepsilon^{(M)}(x) := \widetilde{\Psi}_\varepsilon^{(M)}(x) - \chi(x_2)\big(B_{\varepsilon,\mathrm{D}}^{(M)} + x_2 B_{\varepsilon,\mathrm{N}}^{(M)}\big).$$

In view of lemma 5.4, this function belongs to the domain of $\mathcal{H}_\varepsilon^{(p)}(0)$ and

$$\big(\mathcal{H}_\varepsilon^{(p)}(0) - \Lambda_\varepsilon^{(M)}\big)\Psi_\varepsilon^{(M)} = f_\varepsilon^{(M)}, \qquad \left\|f_\varepsilon^{(M)}\right\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^{M-\frac{3}{2}}|(\ln\eta|^{M-2}+1). \tag{5.12}$$

It is also easy to check that

$$\left\|\Psi_\varepsilon^{(M)}\right\|_{L_2(\Omega_\varepsilon)} = \varepsilon^{1/2}\left(\frac{\pi}{\sqrt{2}} + \mathcal{O}\big(\varepsilon(|\ln\eta|+1)\big)\right).$$

Denote

$$\widehat{\Psi}_\varepsilon^{(M)} := \frac{\Psi_\varepsilon^{(M)}}{\left\|\Psi_\varepsilon^{(M)}\right\|_{L_2(\Omega_\varepsilon)}},$$

and by (5.12) and (4.4) we have

$$\frac{1}{\Lambda_\varepsilon^{(M)}}\widehat{\Psi}_\varepsilon^{(M)} - \left(\mathcal{H}_\varepsilon^{(p)}(0)\right)^{-1}\widehat{\Psi}_\varepsilon^{(M)} = \widehat{f}_\varepsilon^{(M)}, \qquad \left\|\widehat{f}_\varepsilon^{(M)}\right\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^{M-2}|(\ln\eta|^{M-2}+1).$$

Since the operator $\left(\mathcal{H}_\varepsilon^{(p)}(0)\right)^{-1}$ is self-adjoint and compact, by [OSI, chapter III, section 1, lemma 1.1] we conclude that there exists an eigenvalue λ_ε of the operator $\mathcal{H}_\varepsilon^{(p)}(0)$ such that

$$\left|\frac{1}{\Lambda_\varepsilon^{(M)}} - \frac{1}{\lambda_\varepsilon}\right| \leqslant C\varepsilon^{M-2}(|\ln\eta|^{M-2}+1).$$

In view of the asymptotics (5.2), the only eigenvalue of $\mathcal{H}_\varepsilon^{(p)}(0)$ which satisfies this inequality is $\lambda_1(0,\varepsilon)$ and $\lambda_\varepsilon = \lambda_1(0,\varepsilon)$. Hence,

$$\left|\Lambda_\varepsilon^{(M)} - \lambda_\varepsilon\right| \leqslant C\varepsilon^{M-2}(|\ln\eta|^{M-2}+1),$$

and the asymptotics (2.6) is proven.

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